WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #6 due 11/27/2015

Problem 1. a.) Show that the Dirichlet problem for the Laplace operator in $\Omega \subset \mathbb{R}^d$ satisfies the Lopatinskii condition. Prove $\dim\ker T = 0$. (Here T is the operator defined in (2.6.1) in the Lecture Notes.)

Solution. For the Laplacian we know that $P(x,\xi) = \xi \cdot \xi$. Following Corollary 2.7.6 from the lecture notes, consider the ordinary differential equation

$$P\left(x, \xi + in(x)\frac{d}{dy}\right)\Phi(y) = \left(\xi + in(x)\frac{d}{dy}\right)\cdot\left(\xi + in(x)\frac{d}{dy}\right)\Phi(y) = \left[|\xi|^2 - \frac{d^2}{dy^2}\right]\phi(y) = 0\ ,$$

where $\xi \perp n(x)$, $\xi \neq 0$. The bounded solution on the interval $[0,\infty)$ is $\Phi(y) = Ce^{-|\xi|y}$. The Dirichlet boundary condition translates into $\Phi(0) = \eta$. This initial value problem is uniquely solvable and the solution is $\Phi(y) = \eta e^{-|\xi|y}$.

Suppose that $u \in \ker T$. Due to Corollary 2.6.9 we know that $u \in C^\infty(\overline{\Omega})$. Then, using integration by parts gives

$$0 = \int_\Omega u\Delta u\, dx = -\int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u\frac{\partial u}{\partial n} dS\ .$$

Since $u = 0$ on $\partial\Omega$, one has $\nabla u = 0$ and u has to be constant. Because of the boundary condition, one infers that $u \equiv 0$.

b.) Show that the Neumann problem for the Laplace operator in $\Omega \subset \mathbb{R}^d$ satisfies the Lopatinskii condition. Prove $\dim\ker T = 1$.

Solution. Using Corollary 2.7.6 the Neumann boundary condition translates into

$$B\left(x, \xi + in(x)\frac{d}{dy}\right)\Phi(y)\Big|_{y=0} = n(x)\cdot\left(\xi + in(x)\frac{d}{dy}\right)\Phi(y)\Big|_{y=0} = i\Phi'(0)\ .$$

The initial value problem with the initial condition $\Phi'(0) = \eta$ is uniquely solvable with $\Phi(y) = -\eta/|\xi|e^{-|\xi|y}$. To find the kernel of the operator T, one proceeds as in part a.) and concludes that u has to be constant. Thus $\dim\ker T = 1$.

Problem 2. Consider the 4×4 first-order differential operator in $\mathbb{R}^3_+$

$$P(\partial)u = \begin{bmatrix} \nabla\times v + \nabla w \\ -\nabla\cdot v \end{bmatrix}\ .$$

Here $u = \begin{bmatrix} v \\ w \end{bmatrix}$ is a vector-valued function with four components, v is a vector-valued function with three components, and the function w is scalar-valued.

a.) Show that the boundary condition $B_1 u = n \times v$ satisfies the Lopatinskii condition. Here and henceforth $n = -e_3$ is the exterior unit normal vector of $\mathbb{R}^3_+$ along $\partial\mathbb{R}^3_+$.

Solution. Recall from homework #3 that

$$P(\xi) = \begin{bmatrix} 0 & -i\xi_3 & i\xi_2 & i\xi_1 \\ i\xi_3 & 0 & -i\xi_1 & i\xi_2 \\ -i\xi_2 & i\xi_1 & 0 & i\xi_3 \\ -i\xi_1 & -i\xi_2 & -i\xi_3 & 0 \end{bmatrix}$$

and observe that $P(\xi)^2 = |\xi|^2 I_4$. Hence, we know that $\det P(\xi) = \pm[|\xi|^2]^2$. Hence, the bounded solutions to the system

$$P\left(\xi_1, \xi_2, \frac{1}{i}\frac{d}{dy}\right)\varphi(y) = 0 \tag{1}$$

are found as follows. For ξ_1, ξ_2 fixed and $\xi_2^2 + \xi_2^2 \neq 0$ set $\varphi(y) = e^{\lambda y} z$ where $z \in \mathbb{C}^4 \setminus \{0\}$. Then the differential equation (1) gives

$$P\left(\xi_1, \xi_2, \frac{1}{i}\lambda\right) z = 0$$

which implies that $\det P(\xi_1, \xi_2, \lambda/i) = 0$. This determinant is known and we obtain the characteristic equation

$$[\xi_1^2 + \xi_2^2 - \lambda^2]^2 = 0$$

which gives $\lambda = \pm\sqrt{\xi_1^2 + \xi_2^2}$ with algebraic multiplicity two. Since $\det P(\xi_1, \xi_2, -\sqrt{\xi_1^2 + \xi_2^2}/i) = 0$ there is at least one non-zero vector $z \in \mathbb{C}^4$ such that

$$P\left(\xi_1, \xi_2, -\sqrt{\xi_1^2 + \xi_2^2}/i\right) v = 0$$

With $\zeta = (\xi_1, \xi_2, -\sqrt{\xi_1^2 + \xi_2^2}/i)^T = (\xi_1, \xi_2, i\sqrt{\xi_1^2 + \xi_2^2})^T$ one can write

$$P\left(\xi_1, \xi_2, i\sqrt{\xi_1^2 + \xi_2^2}\right) z = P(\zeta) z = \begin{bmatrix} \zeta\times & \zeta \\ \zeta^T & 0 \end{bmatrix} z = 0$$

and one finds

$$z = \begin{bmatrix} \zeta \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ i\sqrt{\xi_1^2 + \xi_2^2} \\ 0 \end{bmatrix}.$$

With trial and error one finds a second linearly independent vector $y \in \mathbb{C}^4$ with

$$y = \begin{bmatrix} \xi_2 \\ -\xi_1 \\ 0 \\ -i\sqrt{\xi_1^2 + \xi_2^2} \end{bmatrix} \quad \text{such that} \quad P(\zeta) y = 0\,.$$

Hence, the exponentially decaying solutions on $\mathbb{R}_+$ of $P\left(\xi_1, \xi_2, \frac{1}{i}\frac{d}{dy_3}\right)\varphi(y) = 0$ are

$$\varphi(y) = \left\{ c_1 \begin{bmatrix} \xi_1 \\ \xi_2 \\ i\sqrt{\xi_1^2 + \xi_2^2} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \xi_2 \\ -\xi_1 \\ 0 \\ -i\sqrt{\xi_1^2 + \xi_2^2} \end{bmatrix} \right\} e^{-\sqrt{\xi_1^2+\xi_2^2}y}\,. \tag{2}$$

Observe that $B_1 u = n \times v = (v_2, -v_1)^T$. We will show that this boundary condition satisfies the Lopatinskii condition. Let $(\varphi_1, \varphi_2)(0) = \eta\mathbb{C}^2$. Then

$$c_1\xi_1 + c_2\xi_2 = \eta_1$$
$$c_1\xi_2 - c_2\xi_1 = \eta_2$$

This system has a unique solution,

$$c_1 = \frac{\eta_1\xi_1 + \eta_2\xi_2}{\xi_1^2 + \xi_2^2}\,, \qquad c_2 = \frac{\eta_1\xi_2 - \eta_2\xi_1}{\xi_1^2 + \xi_2^2}\,.$$

b.) Show that the boundary condition $B_2 u = (n \cdot v, w)$ satisfies the Lopatinskii condition.

Solution. Up to the very end, this problem is solved as part a.). However, note that $B_2\varphi = (\varphi_3(0), \varphi_4(0))$. In this case one has

$$c_1 = \frac{\eta_1}{i\sqrt{\xi_1^2 + \xi_2^2}}\,, \qquad c_1 = \frac{i\eta_2}{\sqrt{\xi_1^2 + \xi_2^2}}\,.$$

c.) Do these two boundary condition satisfy the Lopatinskii condition with respect to the operator

$$P_\alpha(x, \partial)u = \begin{bmatrix} \nabla \times v + \alpha(x)\nabla w \\ -\nabla \cdot (\alpha(x)v) \end{bmatrix}$$

already considered in Homework #3? Suppose that α is a 3×3 Hermitian matrix.

Solution. Yes, this is also true as long as α is positive definite. However, now one needs to find the roots in λ for the quadratic equation

$$(\xi_1, \xi_2, \lambda/i)^T \alpha(\xi_1, \xi_2, \lambda/i)$$

which has one root with negative real part. In this connection it is significant that α is Hermitian and positive definite. The solution is similar to (2) but will involve also the components of α.

Problem 3. Can you find a boundary condition for the system in Problem 2 that does not satisfy the Lopatinskii condition ?

If one prescribes only one scalar boundary condition, then the Lopatinskii condition cannot hold. Two scalar conditions are needed to determined the constants c_1 and c_2 uniquely.